Spin-Hall conductivity due to Rashba spin-orbit interaction in disordered systems

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We consider the spin-Hall current in a disordered two-dimensional electron gas in the presence of Rashba spin-orbit interaction. We derive a generalized Kubo-Greenwood formula for the spin-Hall conductivity σ^z_{yx} and evaluate it in an systematic way using standard diagrammatic techniques for disordered systems. We find that in the diffusive regime both Boltzmann and the weak localization contributions to σ^z_{yx} are of the same order and vanish in the zero frequency limit. We show that the uniform spin current is given by the total time derivative of the magnetization from which we can conclude that the spin current vanishes exactly in the stationary limit. This conclusion is valid for arbitrary spin-independent disorder, external electric field strength, and also for interacting electrons.

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I. INTRODUCTION

The idea of using the spin degrees of freedom of electrons in semiconductor systems instead of their charge has attracted wide interest in recent years[1,2,3,4]. Of particular interest are systems with spin-orbit interaction as this allows access to spin via charge and vice versa. A number of effects in such systems have been studied theoretically[5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22], and experimentally[1,2,3,4,23,24], and most recently by Kato *et al.*[25] who report the first observation of spin Hall effect in GaAs and InGaAs samples of the extrinsic type[5,6]. In particular, there has been considerable interest in Rashba spin-orbit interaction[26] in two-dimensional electron systems since this type of spin-charge coupling can be controlled by electrical gates. Moreover, it has been pointed out by Sinova *et al.*[13] that clean (without impurities) systems with spin-orbit interaction carry spin currents and show a spin-Hall effect: applying an electric field E_x in x-direction generates a spin current in perpendicular y-direction. In a number of subsequent papers the corresponding spin-Hall conductivity σ^z_{yx} has been calculated, and there seems to be agreement now that the spin-Hall conductivity at zero frequency vanishes in an infinite system[12,19,21,27,28,29]. In this paper we present a systematic calculation of σ^z_{yx} by using well-known perturbative techniques for disordered systems[30]. The systematic expansion is performed in powers of the small parameter $1/p_{\rm F}l$, with $p_{\rm F}$ the Fermi momentum and l the mean free path. In addition to confirming the results of earlier work[12,19,21,27,29], which were obtained in the Boltzmann (semiclassical) regime, we go beyond this limit and also include the weak localization correction diagrams. We show that this latter contribution is of the same order in $1/p_{\rm F}l$ and Rashba amplitude as the Boltzmann value and vanishes as well at zero frequency. This demonstration requires us to consider the various contributions to the Hikami box in the weak localization contribution as well as the renormalization of the current vertices.

We then go on and show that the vanishing of the spin current is expected on rather general grounds and under general conditions, such as arbitrary (spin-independent) disorder, electric field strength, and for interacting electrons. Crucial for our argument are two observations. First, the uniform spin current is given by the total time derivative of the magnetization[20], which, in turn, provides a straightforward interpretation of the spin current and its observable effect in terms of magnetization and its change in time. Second, a system driven by an external force which is constant in time normally reaches a steady state, implying then that the spin current vanishes under rather general conditions.

II. MODEL SYSTEM

We consider non-interacting electrons of mass m and charge e moving in a disordered two-dimensional system in the presence of Rashba spin-orbit interaction with amplitude α[13,18,26]. This system is described by the Hamiltonian (we set $\hbar = 1$)

$$\hat{H} = \frac{\hat{p}^2}{2m} + \alpha(\hat{\sigma}^1\hat{p}_y - \hat{\sigma}^2\hat{p}_x) + U(\mathbf{r}), \tag{1}$$

where $\hat{p}$ denotes the momentum operator, and $\hat{\sigma}^\mu$ are the Pauli matrices. Further, $U(\mathbf{r})$ is a short-ranged disorder potential with the property that $\overline{U(\mathbf{r})U(\mathbf{r}\,')} = (m\tau)^{-1}\delta(\mathbf{r}-\mathbf{r}\,')$, where the overbar indicates average over the disorder configuration, and τ the mean free time between collisions.

The spin-orbit term in the Hamiltonian (1) changes the expression for the charge-current density operator (in standard second quantization notation)[20]:

$$\hat{\mathbf{j}}(\mathbf{r}) = \frac{ie}{2m}\left[\left(\nabla\hat{\psi}^\dagger\right)\hat{\psi} - \hat{\psi}^\dagger\nabla\hat{\psi}\right] - \frac{e^2}{mc}\hat{\psi}^\dagger\left(\mathbf{A} + \tilde{\mathbf{A}}\right)\hat{\psi}, \tag{2}$$

where together with an electromagnetic vector potential $\mathbf{A}$ (coupling to the charge e) a fictitious spin-dependent

vector potential $\tilde{\mathbf{A}}$ is introduced:

$$\tilde{\mathbf{A}} = -(\alpha mc/e)(-\hat{\sigma}^2, \hat{\sigma}^1, 0), \tag{3}$$

where c is the speed of light. On the other hand, the spin-current density operator (z-component of spin) remains the same as in the case without spin-orbit interaction[20,31]:

$$\hat{\mathbf{j}}^{s_z}(\mathbf{r}) = -\frac{i}{4m}\Big[\hat{\psi}^\dagger\hat{\sigma}^3\nabla\hat{\psi} - (\nabla\hat{\psi}^\dagger)\hat{\sigma}^3\hat{\psi} \\ -\frac{2ie}{c}\mathbf{A}\hat{\psi}^\dagger(\mathbf{r})\hat{\sigma}^3\hat{\psi}(\mathbf{r})\Big]. \tag{4}$$

III. SPIN-HALL CURRENT

Considering now a linear response regime for the spin current in the presence of an electric field E_x, we derive a generalized Kubo-Greenwood formula for the spin-Hall conductivity. In the Keldysh approach[32] the spin current can be expressed as

$$\overline{\mathbf{j}^{s_z}(\omega)} = \\ -\frac{i}{2m}\int_{-\infty}^{\infty}\frac{\mathrm{d}E}{2\pi}\overline{\mathrm{Tr}\left\{\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)\frac{\hat{\sigma}^3}{2}\hat{G}_\mathrm{K}(E, E-\omega)\right\}}, \tag{5}$$

where G_K is the Keldysh component of the 2x2 matrix Green function; Tr stands for the trace in both momentum and spin space. Due to the Pauli matrix $\hat{\sigma}^3$ the diamagnetic term $\propto \mathbf{A}$ gives a vanishing contribution[33] to the spin current in eq. (5). Then in first order perturbation in $\mathbf{A}$ we find

$$\overline{\delta\hat{G}_\mathrm{K}(E, E-\omega)} = (h_E - h_{E-\omega}) \\ \times\overline{\hat{G}^E_\mathrm{R}\left[-\frac{e}{mc}\left(\hat{\mathbf{p}} - \frac{e}{c}\tilde{\mathbf{A}}\right)\mathbf{A}(\omega)\right]\hat{G}^{E-\omega}_\mathrm{A}}, \tag{6}$$

where $h_E = \tanh(E/2k_\mathrm{B}T)$, with $k_\mathrm{B}T \ll E_\mathrm{F}$ the temperature and E_F the Fermi energy, and where $\hat{G}^E_\mathrm{R/A} = (E - \hat{H} + E_\mathrm{F} \pm i0^+)^{-1}$ are the retarded and advanced Green functions, respectively. From eqs. (5) and (6) we obtain a generalized Kubo-Greenwood formula for the spin-Hall conductivity[34],

$$\sigma^z_{yx}(\omega) = \frac{e}{2\pi m^2}\overline{\mathrm{Tr}\left[\frac{\hat{\sigma}^3}{2}\hat{p}_y\hat{G}^E_\mathrm{R}\left(\hat{p}_x - \frac{e}{c}\tilde{A}_x\right)\hat{G}^{E-\omega}_\mathrm{A}\right]}, \tag{7}$$

where $E, \omega \ll E_\mathrm{F}$. In deriving (7) we have assumed that the electric field $E_x(\omega) = i\omega A_x(\omega)/c$ is applied along the x-direction producing a perpendicular spin current along y-direction, i.e.,

$$\overline{j^{s_z}_y(\omega)} = \sigma^z_{yx}(\omega)E_x(\omega). \tag{8}$$

Next, after averaging over the random disorder, the spin-dependent Green functions $\hat{G}_\mathrm{R/A}$ become diagonal in momentum representation[35], $\overline{\langle\mathbf{p}|\hat{G}^E_\mathrm{R/A}|\mathbf{p}'\rangle} = (2\pi)^2 G^E_\mathrm{R/A}(\mathbf{p})\delta(\mathbf{p} - \mathbf{p}')$, with

$$G^E_\mathrm{R/A}(\mathbf{p}) = \frac{1}{2}\sum_{s=\pm1}\frac{\hat{\sigma}^0 + s\hat{M}}{E - \xi(p) - s\alpha p \pm \frac{i}{2\tau}}, \tag{9}$$

where $\hat{M} = (p_y\hat{\sigma}^1 - p_x\hat{\sigma}^2)/p$ with $\hat{M}^2 = \hat{\sigma}^0 \equiv 1$, $\xi(p) = p^2/2m - E_\mathrm{F}$. The expression (9) has been obtained in the self-consistent Born approximation.

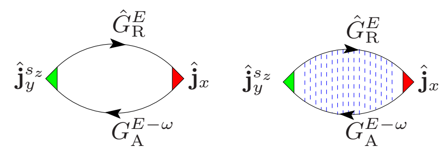


FIG. 1: Diagrams for the spin-Hall conductivity in the zero-loop approximation: The right diagram represents the vertex correction. Dashed lines denote averaging over the impurities.

The diagrams of the standard perturbative approach[30] can be generated from (7) by expanding in the disorder potential U and "dressing" the Green function lines with diffusons and Cooperons. In this formalism[30,36], the resulting diagrams can be estimated according to the number of loops, composed by Cooperon and diffuson lines (loop expansion). In the following sections we will expand σ^z_{yx} in powers of $1/p_\mathrm{F}l$, where $l = v_\mathrm{F}\tau$ denotes the mean free path, and evaluate it up to order $\frac{|e|}{(p_\mathrm{F}l)^0}$, that is, we will neglect all terms with $n \geq 1$ in the expansion

$$\sigma^z_{yx} = |e|\sum_n \frac{s_n}{(p_\mathrm{F}l)^n}, \quad p_\mathrm{F}l \gg 1. \tag{10}$$

Thus in our diagrammatic loop expansion we have to consider two classes of diagrams: (i) diagrams with no loops (semiclassical approximation), and (ii) diagrams with one loop (weak localization corrections). This is so because of the non-standard situation arising from the spin dependence of the charge current vertex in (7), which can be written as

$$\hat{p}_x - \frac{e}{c}\tilde{A}_x = p_\mathrm{F}\hat{n}_x + \left(\hat{p}_x - p_\mathrm{F}\hat{n}_x - \frac{e}{c}\tilde{A}_x\right), \tag{11}$$

where p_F is the Fermi momentum, and $\hat{\mathbf{n}} = \hat{\mathbf{p}}/p$ is an operator of the direction of momentum $\hat{\mathbf{p}}$. The contribution in the brackets on the right-hand side of (11) is a correction of order $(p_\mathrm{F}l)^{-1} \ll 1$ to the main term $p_\mathrm{F}\hat{n}_x$. Thus, in zero-loop approximation (see Sec. IV) this correction leads to a contribution which is of the same order as the one coming from $p_\mathrm{F}\hat{n}_x$ in the one-loop approximation (i.e. weak localization correction – see Sec. V). Hence, both of them are expected to give contributions of the same order in $1/p_\mathrm{F}l$ to the spin-Hall conductivity. The spin current vertex $\hat{\sigma}_z\hat{p}_y$, on the other hand, can always be approximated by $\hat{\sigma}_z p_\mathrm{F}\hat{n}_y$ in the order considered in this work.

IV. ZERO-LOOP APPROXIMATION

In zero-loop approximation we need to retain only the diagrams depicted in Fig. 1 with no loops. This approximation corresponds to the semiclassical (Boltzmann) limit. To calculate the first diagram, we simply substitute $\hat{G}_{\mathrm{A/R}}$ in (7) by the averaged Green functions (9). One can see that the main term of the charge current vertex (11) does not contribute in the first diagram. The remaining terms in the brackets in (11) give

$$\sigma_{yx}^{z(0)}(\omega) = \frac{|e|}{8\pi} \times \frac{x^2}{x^2+\lambda^2}, \quad \lambda = 1 - i\omega\tau, \tag{12}$$

where $x \equiv 2\alpha p_{\mathrm{F}}\tau$ is the dimensionless spin-orbit parameter which measures the spin rotation between two consecutive impurity collisions. For $\omega = 0$ this expression agrees with the qualitative result of Ref.[18] in the limit of small spin-orbit interaction $m\alpha^2\tau \ll 1$, as well as in the ballistic limit $\tau \to \infty$.

Next we turn to the vertex correction, described by the second diagram in Fig. 1. Again, it is crucial[37] to retain the entire correction of the charge current vertex (11) (which amounts to include curvature effects of the Fermi surface), since it will turn out now that this correction term leads to an exact cancellation of $\sigma_{yx}^{z(0)}$ given in eq. (12). We note that the scattering off impurities is spin-independent, so that dashed lines do not alter the spin direction which simplifies the calculation considerably. For further evaluation it is convenient to apply the following identity for the Pauli matrices[38]

$$\hat{\sigma}^0_{s_1 s_2}\hat{\sigma}^0_{s_3 s_4} = \frac{1}{2}\sum_{\mu=0}^{3}\hat{\sigma}^{\mu}_{s_1 s_3}\hat{\sigma}^{\mu}_{s_4 s_2} \tag{13}$$

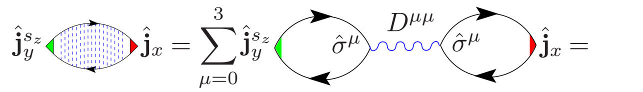
to every dashed line in the vertex diagram Fig. 1, with $\alpha, \beta = 0, \ldots, 3$, and where, again, $\hat{\sigma}^0$ is the unity matrix. This trick reduces the vertex diagram to a geometric series consisting of terms where the momentum and spin summation can be performed separately. These terms are given by

$$X_{\mathrm{D}}^{\alpha\beta} = \frac{1}{2m\tau}\,\mathrm{Tr}[\hat{\sigma}^{\alpha}G_{\mathrm{R}}^{E}(\mathbf{p})\hat{\sigma}^{\beta}G_{\mathrm{A}}^{E-\omega}(\mathbf{p})], \tag{14}$$

where from now on Tr stands for $\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\,\mathrm{Tr}_{\mathrm{spin}}$. The off-diagonal elements of $X_{\mathrm{D}}^{\alpha\beta}$ vanish, while the diagonal ones are given by

$$X_{\mathrm{D}}^{00} = \frac{1}{\lambda}, \quad X_{\mathrm{D}}^{33} = \frac{\lambda}{x^2+\lambda^2},$$
$$X_{\mathrm{D}}^{11} = X_{\mathrm{D}}^{22} = \frac{1}{2}\left[\frac{1}{\lambda} + \frac{\lambda}{x^2+\lambda^2}\right]. \tag{15}$$

The expression for the vertex diagram can be split into three parts, and the vertex diagram can be represented by two "bubbles" with a diffuson wavy line in between:



$$= \frac{e}{2\pi m^2}\sum_{\mu=0}^{3}\mathrm{Tr}\left[\frac{\hat{\sigma}^3 p_y}{2}G_{\mathrm{R}}^{E}(\mathbf{p})\hat{\sigma}^{\mu}G_{\mathrm{A}}^{E-\omega}(\mathbf{p})\right]\times$$
$$\times D^{\mu\mu}\,\mathrm{Tr}\left[\hat{\sigma}^{\mu}G_{\mathrm{R}}^{E}(\mathbf{p})\left(p_x - \frac{e}{c}\tilde{A}_x\right)G_{\mathrm{A}}^{E-\omega}(\mathbf{p})\right], \tag{16}$$

where $D^{\mu\mu}$ denotes the diffuson

$$D^{\mu\mu} = \frac{1}{2m\tau}\frac{1}{1 - X_{\mathrm{D}}^{\mu\mu}}, \tag{17}$$

with $X_{\mathrm{D}}^{\mu\mu}$ given in (15). Only the term with $\mu = 2$ contributes to (16). The two bubbles in (16) are explicitly given by

$$\mathrm{Tr}\left[\frac{\hat{\sigma}^3 p_y}{2}G_{\mathrm{R}}^{E}(\mathbf{p})\hat{\sigma}^2 G_{\mathrm{A}}^{E-\omega}(\mathbf{p})\right] = \frac{x m\tau p_{\mathrm{F}}/2}{x^2+\lambda^2}, \tag{18}$$

and

$$\mathrm{Tr}\left[\hat{\sigma}^2 G_{\mathrm{R}}^{E}(\mathbf{p})\left(p_x - \frac{e}{c}\tilde{A}_x\right)G_{\mathrm{A}}^{E-\omega}(\mathbf{p})\right] =$$
$$= \frac{\alpha m^2\tau}{\lambda}\frac{x^2}{x^2+\lambda^2}. \tag{19}$$

Inserting these expressions into (16), we finally obtain for the vertex correction of the spin-Hall conductivity

$$\sigma_{yx}^{z(1)}(\omega) = -\frac{|e|}{8\pi}\frac{x^2}{x^2+\lambda^2}\frac{x^2/2}{(1-2i\omega\tau)\frac{x^2}{2} - i\omega\tau\lambda^2}. \tag{20}$$

Adding now eqs. (12) and (20) we obtain the spin-Hall conductivity in zero-loop approximation,

$$\sigma_{yx}^{z}(\omega) = \sigma_{yx}^{z(0)}(\omega) + \sigma_{yx}^{z(1)}(\omega) =$$
$$= \frac{|e|}{8\pi}\frac{x^2}{x^2+\lambda^2}\left\{1 - \frac{x^2/2}{(1-2i\omega\tau)x^2/2 - i\omega\tau\lambda^2}\right\}, \tag{21}$$

which is the same result as obtained in Ref.[19] (see eq. (25) therein) by the equation of motion approach. Thus, in zero-loop approximation the two diagrams in Fig. 1 cancel each other, so that the leading contribution to the spin-Hall conductivity vanishes in the limit $\omega = 0$, in agreement with[12,19,21,27,28,29]. A similar cancelation has been observed also for the case of the electric conductivity[39]. In both these cases the "anomalous" term $\propto \tilde{A}$ in the charge current operator (2) is cancelled by the vertex correction at $\omega = 0$. Thus, the second diagram in Fig. 1 can be interpreted as a renormalization of the charge current vertex which results in the cancellation of the "anomalous" term $\propto \tilde{A}_x$.

V. WEAK LOCALIZATION CORRECTION

At the end of Sec. III we argued that in order to reach a given accuracy, we have to include the contribution

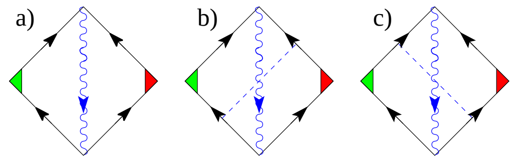


FIG. 2: Weak localization diagrams. The wavy line represents the Cooperon (23). Each diagram contains one spin current vertex (left) and one charge current vertex (right).

of one-loop diagrams in our calculation of σ^z_{yx}. In this section we demonstrate this now by calculating the weak localization correction to σ^z_{yx} explicitly.

Similar to the case of the charge conductivity[40], the weak localization correction to the spin-Hall conductivity is given by three diagrams depicted in Fig. 2. For simplicity, we calculate them at zero frequency $\omega = 0$ only; to simplify notations we omit the energy superscript of Green functions, assuming $G_{\mathrm{R/A}} \equiv G^E_{\mathrm{R/A}}$ below.

Using (13), we may separate the spin indices of a Hikami box from those of the Cooperon, like we did above in (16). Then the total contribution of the three diagrams in Fig. 2 can be written as

$$\sigma^{z(2)}_{yx}(\omega) = \frac{e}{2\pi m^2} \sum_{\gamma,\gamma'=0}^{3} \int \frac{\mathrm{d}^2 q}{(2\pi)^2} C^{\gamma\gamma'}(\mathbf{q}) V^{\gamma\gamma'}(\mathbf{q}), \quad (22)$$

where $C^{\gamma\gamma'}(\mathbf{q})$ are the matrix elements of the Cooperon defined as

$$C^{\gamma\gamma'}(\mathbf{q}) = \frac{1}{2m\tau} \left[\frac{X_{\mathrm{C}}(\mathbf{q})}{1 - X_{\mathrm{C}}(\mathbf{q})} \right]_{\gamma\gamma'}, \gamma, \gamma' = 0 \dots 3, \quad (23)$$

$$X^{\alpha\beta}_{\mathrm{C}}(\mathbf{q}) = \frac{1}{2m\tau} \mathrm{Tr}[\hat{\sigma}^\alpha G_{\mathrm{R}}(\mathbf{p}) \hat{\sigma}^\beta (G_{\mathrm{A}}(\mathbf{q} - \mathbf{p}))^T]. \quad (24)$$

In eq. (22), $V^{\gamma\gamma'}(\mathbf{q})$ denotes the Hikami box:

$$V^{\gamma\gamma'}(\mathbf{q}) = V^{\gamma\gamma'}_a(\mathbf{q}) + V^{\gamma\gamma'}_b(\mathbf{q}) + V^{\gamma\gamma'}_c(\mathbf{q}). \quad (25)$$

Each of the three terms in eq. (25) corresponds to one diagram in Fig. 2 plus the corresponding diagram in Fig. 3 (see eq. (36) below).

In the rest of this section we assume $x^2 \ll 1$. This permits us to approximate (22) by

$$\sigma^{z(2)}_{yx}(\omega) = \frac{e}{2\pi m^2} \sum_{\gamma=0}^{3} V^{\gamma\gamma}(0) \int_{0<q<1/l} \frac{\mathrm{d}^2 q}{(2\pi)^2} C^{\gamma\gamma}(\mathbf{q}). \quad (26)$$

Here we have neglected the off-diagonal elements of the Cooperon, as well as the $q \neq 0$ corrections to the Hikami box in (25). This is justified since those terms do not contain logarithms in x (or dephasing length), in contrast to the diagonal Cooperon contributions in (26) [see eqs. (38),(39) below].

For $q = 0$, the three contributions to the Hikami box in eq. (25) become

$$V^{\gamma\gamma}_a = \mathrm{Tr} \left\{ L(\mathbf{p}) \hat{\sigma}^\gamma \left[\hat{\sigma}^\gamma R(-\mathbf{p}) \right]^T \right\}, \quad (27)$$

$$V^{\gamma\gamma}_{b(c)} = \frac{1}{2m\tau} \sum_{\mu=0}^{3} A^{\gamma\mu}_{b(c)} B^{\mu\gamma}_{b(c)}, \quad (28)$$

where for the left (L) and right (R) vertex parts we use the notations

$$L(\mathbf{p}) = G_{\mathrm{A}}(\mathbf{p}) p_y \frac{\hat{\sigma}^3}{2} G_{\mathrm{R}}(\mathbf{p}), \quad (29)$$

$$\begin{aligned} R(-\mathbf{p}) = G_{\mathrm{R}}(-\mathbf{p}) \left(-p_x - \frac{e}{c} \tilde{A}_x \right) G_{\mathrm{A}}(-\mathbf{p}) \approx \\ \approx G_{\mathrm{R}}(-\mathbf{p}) \left(-n_x p_{\mathrm{F}} \right) G_{\mathrm{A}}(-\mathbf{p}). \end{aligned} \quad (30)$$

Retaining only leading order, we have approximated the charge current vertex in (30) in accordance with the discussion at the end of Sec. III. [The same approximation is made in the calculation of the weak localization correction to the charge conductivity[41].] The quantities A and B in (28) are defined as

$$A^{\gamma\mu}_b = \mathrm{Tr} \left\{ L(\mathbf{p}) \hat{\sigma}^\gamma G^T_{\mathrm{A}}(-\mathbf{p}) \hat{\sigma}^\mu \right\}, \quad (31)$$

$$B^{\mu\gamma}_b = \mathrm{Tr} \left\{ \hat{\sigma}^\gamma R(-\mathbf{p}) \left[G_{\mathrm{A}}(\mathbf{p}) \hat{\sigma}^\mu \right]^T \right\}, \quad (32)$$

$$A^{\gamma\mu}_c = \mathrm{Tr} \left\{ L(\mathbf{p}) \hat{\sigma}^\mu \left[\hat{\sigma}^\gamma G_{\mathrm{R}}(-\mathbf{p}) \right]^T \right\}, \quad (33)$$

$$B^{\mu\gamma}_c = \mathrm{Tr} \left\{ \hat{\sigma}^\mu G_{\mathrm{R}}(\mathbf{p}) \hat{\sigma}^\gamma \left[R(-\mathbf{p}) \right]^T \right\}. \quad (34)$$

Due to the symmetry properties

$$\begin{aligned} (A^{2\mu}_b)^* &= -A^{2\mu}_c, \quad (B^{\mu 2}_b)^* = -B^{\mu 2}_c, \\ (A^{\gamma\mu}_b)^* &= A^{\gamma\mu}_c, \quad (B^{\mu\gamma}_b)^* = B^{\mu\gamma}_c, \quad \gamma \neq 2, \end{aligned} \quad (35)$$

we see that for all γ $V^{\gamma\gamma}_b = (V^{\gamma\gamma}_c)^*$ in (28).

In addition, for every diagram in Fig. 2 one needs to consider the corresponding renormalization of the spin current vertices, which turns out to be of the same order in $(p_{\mathrm{F}} l)^{-1}$ as the bare $L(\mathbf{p})$,

$$\begin{aligned} \tilde{L}(\mathbf{p}) = L(\mathbf{p}) + \sum_{\mu=0}^{3} G_{\mathrm{A}}(\mathbf{p}) \hat{\sigma}^\mu G_{\mathrm{R}}(\mathbf{p}) \times \\ \times D^{\mu\mu} \, \mathrm{Tr} \left\{ \hat{\sigma}^\mu L(\mathbf{p}) \right\}. \end{aligned} \quad (36)$$

The corresponding diagrams are depicted in Fig. 3. Their contributions add to the ones of the bare diagrams in Fig. 2. Note that the symmetry properties (35) remain valid when the renormalization (36) is taken into account in the expressions for A and B.

Similary, the charge current vertex ($\propto R(\mathbf{p})$) gets renormalized, but unlike before this renormalization is of higher order in $(p_{\mathrm{F}} l)^{-1}$ [see at the end of Sec. IV], and hence can be neglected. Still, we note that like in

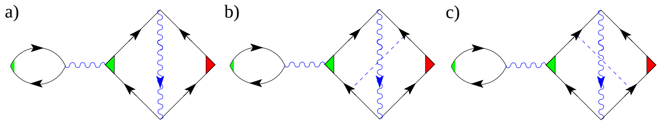


FIG. 3: The weak localization diagrams from Fig. 2 with the renormalized spin current vertex ($\propto L(\mathbf{p})$), see eq. (36). The analogous renormalization of the current vertex ($\propto R(\mathbf{p})$) is of higher order in $(p_{\rm F}l)^{-1}$ and can be neglected.

Sec. IV one can demonstrate that this renormalization of $R(\mathbf{p})$ results in cancellation of the "anomalous" term $\propto \tilde{A}_x$ in the charge current vertex in (30).

Now we have to obtain an expression for $\int \frac{\mathrm{d}^2q}{(2\pi)^2} C^{\gamma\gamma}(\mathbf{q})$ in (26). For this we can make use of the results of Ref.[35], where an analogous approach has been used to study the weak localization correction to the charge conductivity in the same model system. The Cooperon derived in Ref.[35] is connected with C in eq. (23) via

$$C^{\gamma\gamma'}(\mathbf{q}) = \frac{1}{4}\sum_{\alpha,\beta,\mu,\lambda=1}^{2} C^{\alpha\lambda}_{\beta\mu}\hat{\sigma}^{\gamma}_{\beta\alpha}\hat{\sigma}^{\gamma'}_{\lambda\mu}, \tag{37}$$

where $C^{\alpha\lambda}_{\beta\mu}$ is the Cooperon from Ref.[35]. Then, from eqs. (13) and (14) in Ref.[35] we obtain that

$$\int \frac{\mathrm{d}^2q}{(2\pi)^2} C^{\gamma\gamma}(\mathbf{q}) = \frac{2}{\pi l^2 m\tau}(f, f, \log\frac{L_\phi}{l}, f), \tag{38}$$

where[35] L_φ stands for the dephasing length, and

$$f = \begin{cases} \ln\frac{L_\varphi}{l}, & \text{for} \quad x \ll \frac{l}{L_\varphi}, \\ \ln\frac{1}{x}, & \text{for} \quad \frac{l}{L_\varphi} \ll x \ll 1. \end{cases} \tag{39}$$

We note that due to this logarithmic dependence on x the Cooperon itself can become large for sufficiently small x and large dephasing length L_ϕ. However, this has no consequences for the spin Hall conductivity eq. (26), since the Hikami box contributions add up to zero, as we shall see next. Indeed, after some lengthy but straightforward calculation we find for the diagonal terms of the Hikami box[42]

$$V^{\gamma\gamma}_a = (0,0,0,0) + O(x^4), \tag{40}$$

$$\begin{aligned} V^{\gamma\gamma}_b &= -V^{\gamma\gamma}_c \\ &= -\frac{imp_{\rm F}^2\tau^3}{16}(x^2, 8-15x^2, -x^2, 8-11x^2). \end{aligned} \tag{41}$$

Quite remarkably, the terms independent of x come from the vertex renormalization (2nd term on the rhs of eq. (36)), where the only non-zero contribution arises from the diffuson matrix element $D^{22} \propto 1/x^2$. Each of the terms V_b and V_c, when inserted into eq. (26), gives a contribution to $\sigma^{z(2)}_{yx} \propto e$, which corresponds to the term $n = 0$ in (10). Adding up all three terms of the Hikami box we obtain $V^{\gamma\gamma}(0) = 0$ in (25). Thus, we see that the weak localization contribution to the spin Hall conductivity, i.e. $\sigma^{z(2)}_{yx}$ (see eq. (26)), vanishes in the limit $\omega = 0$.

The evaluation of the zero- and one-loop diagrams performed in this and the preceding sections is valid up to the order $1/p_{\rm F}l$. If one would like to go beyond that order (which is not attempted here), many more corrections (e.g. of the type $1/E_{\rm F}\tau$) and many more diagrams had to be taken into account. For instance, one would also need to go beyond the self-consistent Born approximation for the self-energy and to retain crossing diagrams. Of course, the weak localization contribution which includes the Cooperon (resulting from time-reversed paths) is special in the sense that (apart form the log-dependence) it is sensitive to phase coherent effects probed by an Aharonov-Bohm flux. Thus, other contributions, even if they are of the same order in $1/p_{\rm F}l$, could be ignored provided one would be interested in the phase sensitive contributions only.

VI. SPIN CURRENT AND SPIN PRECESSION

We have seen that in perturbation theory the spin-Hall conductivity vanishes in the zero frequency limit, in leading ($\propto (1/p_{\rm F}l)^0$) *and* subleading ($\propto (1/p_{\rm F}l)^1$) order. This result suggests that the vanishing of the spin current is an exact property of the system under consideration.

Indeed, we give now a simple argument to support this claim, based on the generalization of the Hamiltonian (1):

$$\begin{aligned} \hat{H}'(\hat{\mathbf{p}},\mathbf{r}) = \frac{\hat{p}^2}{2m} &+ U(\mathbf{r}) + \alpha\left(\hat{\sigma}_1\hat{p}_y - \hat{\sigma}_2\hat{p}_x\right) + \\ &+ \beta\left(\hat{\sigma}_1\hat{p}_x - \hat{\sigma}_2\hat{p}_y\right) + e\mathbf{r}\mathbf{E}, \end{aligned} \tag{42}$$

where $\mathbf{E}$ is the applied static electric field, and β is the amplitude of Dresselhaus spin-orbit interaction.

From the Heisenberg equation of motion for the spin $\hat{\mathbf{s}} = \hat{\boldsymbol{\sigma}}/2$ of the electron, i.e. $\dot{\hat{\mathbf{s}}}(t) \equiv \frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{s}}(t) = i[\hat{H}', \hat{\mathbf{s}}](t)$, we obtain a relation between spin precession and spin current[43].

$$\begin{aligned} -\frac{1}{2m}\dot{\hat{s}}_x(t) &= \alpha\hat{j}^{s_z}_x(t) + \beta\hat{j}^{s_z}_y(t), \\ -\frac{1}{2m}\dot{\hat{s}}_y(t) &= \alpha\hat{j}^{s_z}_y(t) + \beta\hat{j}^{s_z}_x(t), \end{aligned} \tag{43}$$

or

$$\left.\begin{aligned} \hat{j}_x &= -\frac{1}{2m}\frac{\alpha\dot{\hat{s}}_x - \beta\dot{\hat{s}}_y}{\alpha^2-\beta^2}, \\ \hat{j}_y &= -\frac{1}{2m}\frac{\beta\dot{\hat{s}}_x - \alpha\dot{\hat{s}}_y}{\beta^2-\alpha^2}, \end{aligned}\right. \quad \alpha \neq \beta. \tag{44}$$

The uniform spin current operator (z-component of the spin) is obtained from (4) and given in first quantization

notation by $\hat{\mathbf{j}}^{s_z} = (1/2)\{\hat{\mathbf{v}}, \hat{s}_z\}$, with (spin-dependent) velocity operator $\hat{\mathbf{v}} = i[\hat{H}', \mathbf{r}]$.

From the exact relations (44) we see that the spin current can be experimentally accessed by measuring the spin precession or the change of magnetization in time, which can be done e.g. with optical methods[23,24,25].

In case of $\alpha = 0$ or $\beta = 0$ (44) provides a simple physical interpretation of the spin current: The spatial x (y)-component of the spin current which carries the z-component of the spin is, up to a coupling constant, given by the total time derivative of the x (y)-component of the spin.

Next, let us consider the expectation value (including disorder average) of (44) for $t \to \infty$. The presence of disorder is expected to provide a relaxation mechanism such that the system can reach a steady nonequilibrium state (after some transients) when driven by an external electric field (assumed to be constant in time). The weaker the disorder the longer it takes to reach this stationary limit, but for any finite amount of disorder (finite density of random impurities), it will be reached eventually. In particular, the magnetization $\langle \hat{\mathbf{s}} \rangle(t)$ of such a state also reaches a stationary value (possibly different from zero) and does not depend on time anymore. [Phenomenologically, the approach to such a state is described by a Bloch equation for the spin dynamics; see also below.] Thus, it immediately follows that the rate of magnetization change, $\langle \dot{\hat{\mathbf{s}}} \rangle(t)$, must vanish in such a steady state, and, consequently, the spin current vanishes as well. In other words, under the stated conditions *the uniform spin current vanishes in the long time limit if and only if the magnetization reaches a stationary state,* i.e.

$$\lim_{t\to\infty} \langle \hat{\mathbf{j}}^{s_z} \rangle(t) = 0 \Longleftrightarrow \lim_{t\to\infty} \langle \hat{s}_k \rangle(t) = \text{const.}, \quad k = x, y. \tag{45}$$

The result (45) holds for any form of the static (spin-independent) impurity potential $U(\mathbf{r})$ in the Hamiltonian (42), including the special case of isotropic scattering potentials considered in previous sections. Also, the argument is valid for any strength of the electric field, and, hence, applies to the special case of linear response considered in the previous section [with $\langle \hat{j}_y^{s_z} \rangle \equiv \overline{j_y^{s_z}}$]. Thus, from $0 = \langle \hat{j}_y^{s_z} \rangle(t \to \infty) = \sigma_{yx}^z(\omega = 0) E_x$ we conclude that the spin Hall conductivity $\sigma_{yx}^z(\omega = 0)$ vanishes exactly (i.e. in all orders of the disorder potential) for any finite amount of disorder.

In the special case[44] $\alpha = \beta$, it is convenient to rewrite $\hat{H}'$ in (42) in the coordinate system, rotated counterclockwise by $\pi/4$ in the xy-plane:

$$\begin{aligned} &\hat{H}_{\mathrm{R}}(\hat{\mathbf{p}}', \mathbf{r}') \equiv \hat{H}'(R_{\pi/4}\hat{\mathbf{p}}, R_{\pi/4}\mathbf{r}) = \\ &= \frac{\hat{p}'^2}{2m} - 2\alpha \hat{\sigma}_2' \hat{p}_x' + U'(\mathbf{r}') + e\mathbf{r}'\mathbf{E}' \equiv \hat{H}_{\mathrm{R}0} + e\mathbf{r}'\mathbf{E}', \end{aligned} \tag{46}$$

where

$$R_{\pi/4} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{47}$$

and for convenience we have introduced an alternative set of Pauli matrices

$$\hat{\sigma}_{12}' = \frac{1}{\sqrt{2}}(\hat{\sigma}_2 \pm \hat{\sigma}_1), \quad \hat{\sigma}_3' \equiv \hat{\sigma}_3. \tag{48}$$

Suppose before switching on the electric field (at $t = 0$) the system was described by an equilibrium density matrix $\hat{\rho}_0 = e^{-\hat{H}_{\mathrm{R}0}/T}/Z$. [It implies $\langle \hat{\mathbf{j}}'^{s_z} \rangle(t = 0) = 0$.] Then the spin current in the rotated system is given by

$$\langle \hat{\mathbf{j}}'^{s_z} \rangle(t) = \mathrm{Tr}\left[\frac{\hat{\sigma}_3}{2}\frac{\hat{\mathbf{p}}}{m} e^{-i\hat{H}_{\mathrm{R}}t} \hat{\rho}_0 e^{i\hat{H}_{\mathrm{R}}t}\right]. \tag{49}$$

Since

$$\begin{aligned} &\operatorname*{Tr}_{\mathrm{spin}}[\hat{\sigma}_k' \hat{H}_{\mathrm{R}}] = 0, \quad \operatorname*{Tr}_{\mathrm{spin}}[\hat{\sigma}_k' \hat{H}_{\mathrm{R}0}] = 0, \quad k = 1, 3, \\ &\Longrightarrow \operatorname*{Tr}_{\mathrm{spin}}\left[\frac{\hat{\sigma}_3}{2}\frac{\hat{\mathbf{p}}}{m} e^{-i\hat{H}_{\mathrm{R}}t} \hat{\rho}_0 e^{i\hat{H}_{\mathrm{R}}t}\right] = 0, \quad \forall t, \end{aligned} \tag{50}$$

so that $\langle \hat{\mathbf{j}}^{s_z} \rangle(t) = 0$ for all $t > 0$. Thus, once $\langle \hat{\mathbf{j}}^{s_z} \rangle$ is zero at $t = 0$, it remains zero forever due to the fact that $(\hat{s}_y - \hat{s}_x)$ is a conserved quantity for the Hamiltonian (42).

We consider now $\beta = 0$. The approach of $\langle \dot{\hat{\mathbf{s}}} \rangle(t)$ to its stationary value (in the linear response regime) can be easily illustrated by taking the inverse Laplace transform of eqs. (8) and (21) (restricting ourselves to the Boltzmann value). This involves solving a cubic equation for obtaining the poles with respect to ω. The resulting expression for $\langle \dot{\hat{s}}_y \rangle(t)$ consists of two parts (too lengthy to be written down here), one coming from the real pole and an oscillatory one coming from the pair of complex conjugate poles. For $x^2 \ll 1$ and $t \gg \tau$, only the first part is relevant, which has no oscillations and decays exponentially to zero:

$$\langle \dot{\hat{s}}_y \rangle(t) = -2m\alpha \langle \hat{j}_y^{s_z} \rangle(t) = -\frac{|e|}{4\pi} m\alpha x^2 E_x\, e^{-t/T}, \quad t \gg \tau, \tag{51}$$

where $T/2 = (\Delta^2\tau)^{-1}$ is the well-known Dyakonov-Perel spin relaxation time[45]. For $x^2 \gtrsim 1$ and $t \lesssim \tau$, the oscillatory part in $\langle \dot{\hat{s}}_y \rangle(t)$ becomes dominant, with period $1/\Delta$ and exponential decay with rate $1/\tau$; its time dependence for a particular value of x is illustrated in Fig. 4.

The above consideration can be generalized to the case with electron-electron interaction. In this case, the total spin of the system, $\sum_i \hat{s}_{i,k}$, and the total spin current, $\sum_i \hat{j}_{i,k}^{s_z}$, $k = x, y$, obey the same equation as before, i.e. eq. (51). Thus, the same argument goes through as well, showing that the spin current must vanish in the stationary limit also for interacting systems–provided this limit exists, which, again, we expect to be the case in the presence of any finite amount of disorder.

Finally, in the absence of disorder, no stationary limit of $\langle \hat{\mathbf{s}} \rangle(t)$ can be reached, i.e. the magnetization can change indefinitely, and thus there can be a finite spin current in this very particular case. [We note that the physically observable quantity, the magnetization

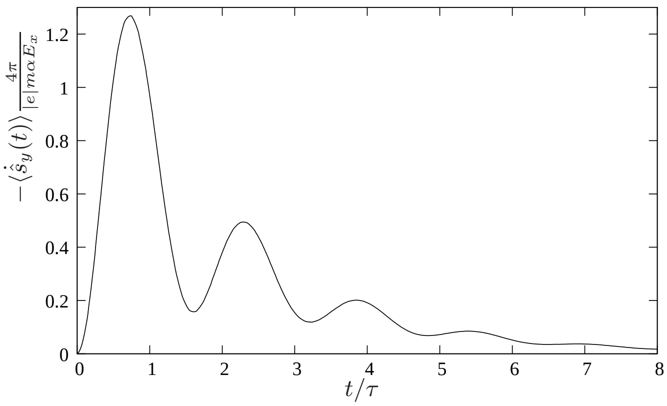


FIG. 4: The time evolution of $\langle \dot{\hat{s}}_y(t) \rangle$ for $x = 4$. The period of oscillation is $1/\Delta$, and the exponential decay time is τ. Note that for any x, $\langle \dot{\hat{s}}_y(t) \rangle = 0$ at $t = 0$.

change, vanishes for vanishing Rashba coupling.] However, when the spin current approaches a non-zero but constant value for $t \to \infty$, as obtained in the linear response regime for a clean system[16], the magnetization $\langle \hat{s}_k \rangle(t)$ actually grows linearly in time in the asymptotic regime $t \to \infty$. This, of course, shows a breakdown of the linear response approximation in this case since the bound for a spin 1/2, i.e. $|\langle \hat{s}_k \rangle(t)| \leq 1/2$, is violated. The terms beyond linear response would restore the bounded and oscillatory behavior of $\langle \hat{s}_k \rangle(t)$.

VII. CONCLUSIONS

In conclusion, we have calculated the Boltzmann and weak localization contributions to the spin-Hall conductivity σ^z_{yx} in an infinite-size 2D disordered system with Rashba spin-orbit interaction. In our calculation we had to consider the weak localization contribution to the σ^z_{yx}, together with the Boltzmann contribution, since diagrams belonging to these two contributions have the same order of magnitude. In the diffusive and static limit, the spin-Hall conductivity vanishes. We have shown that the spin current is given by the time derivative of the magnetization change and have given general arguments why the spin current must vanish for any finite amount of disorder that permits to reach a stationary limit of the system when driven by a constant electric field. It will be interesting to see how far our arguments can be extended to other types of spin orbit interaction.

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